Today

Sorror speed governor for escalators

1. Differential equations
2. Simulation diagrams
3. State equations
A system described by a differential equation is of the form

\[
\begin{align*}
\begin{cases}
    a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y &= G(x(t)) \\
y(t_0) &= y_0, \quad y'(t_0) = y_1, \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}.
\end{cases}
\end{align*}
\]

If \( y_0 = \ldots = y_{n-1} = 0 \) then the system is initially at rest or initially relaxed.

**Theorem**

*If the system is initially at rest and if \( G \) is linear then (*) is an LTI system.*

Let \( y(t) \) be the solution of (*) for arbitrary input \( x(t) \). The system’s transfer function \( H(s) \) is

\[
H(s) = \frac{Y(s)}{X(s)},
\]

where \( X(s) \) is the Laplace transform of \( x(t) \) and \( Y(s) \) is the Laplace transform of \( y(t) \).

**Theorem**

Assume an initially relaxed system is be described by the differential equation

\[
\begin{align*}
\begin{cases}
    a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y &= b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_0 x(t).
\end{cases}
\end{align*}
\]

Define

\[
A(s) = a_n s^n + a_{n-1} s^{n-1} \cdots + a_1 s + a_0
\]

and

\[
B(s) = b_m s^m + b_{m-1} s^{m-1} \cdots + b_1 s + b_0.
\]

then

\[
H(s) = \frac{B(s)}{A(s)}
\]
Example

The system \( L \) satisfies the condition of initial rest and is described by the following differential equation:

\[ y'' - 3y' + 2y = x(t). \]

Find the response of \( L \) to input \( x(t) = r(t) = tu(t) \).

- \( A(s) = s^2 - 3s + 2 \) and \( B(s) = 1 \).
- \( H(s) = B(s)/A(s) = \frac{1}{s^2 - 3s + 2} \).
- \( X(s) = \mathcal{L}\{t\} = \frac{1}{s^2} \).
- \( Y(s) = H(s)X(s) = \frac{1}{s^2(s^2 - 3s + 2)} \)
  \[ = \frac{1}{s^2} - \frac{1}{s - 1} + \frac{3}{4} \frac{1}{s} + \frac{1}{2} \frac{1}{s^2}. \]
- \( y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left(\frac{1}{4}e^{2t} - e^t + \frac{3}{4} + \frac{1}{2}t\right) u(t). \)

Example (continued)

- The response is
  \[ y(t) = \left(\frac{1}{4}e^{2t} - e^t + \frac{3}{4} + \frac{1}{2}t\right) u(t). \]
- The response is continuously differentiable:
  \[ y'(t) = \left(\frac{1}{2}e^{2t} - e^t + \frac{1}{2}\right) u(t) + \left(\frac{1}{4}e^{2t} - e^t + \frac{3}{4} + \frac{1}{2}t\right) \delta(t) \]
  \[ = \left(\frac{1}{2}e^{2t} - e^t + \frac{1}{2}\right) u(t) + \left(\frac{1}{4}e^{2\cdot0} - e^0 + \frac{3}{4} + 0\right) \delta(t) \]
  \[ = \left(\frac{1}{2}e^{2t} - e^t + \frac{1}{2}\right) u(t). \]
  and \( y'(0^+) = 0 \).
- In a similar way you can prove that \( y(t) \) is two times continuously differentiable: \( y''(t) = (e^{2t} - e^t) u(t) \).
Example

The system L is described by the following differential equation:

\[ y'' - 3y' + 2y = x(t). \]

Find the step response (the response to input \( x(t) = u(t) \)).

- \( H(s) = \frac{1}{s^2 - 3s + 2} \).
- \( X(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \).
- \( Y(s) = H(s)X(s) = \frac{1}{s} \frac{1}{s - 2} \frac{1}{s - 1} \).
  \[ = \frac{1}{2s} + \frac{1}{2(s - 2)} - \frac{1}{s - 1}. \]
- \( y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left( \frac{1}{2} + \frac{1}{2} e^{2t} - e^t \right) u(t) \).

Example (continued)

- The step response is
  \[ y(t) = \left( \frac{1}{2} + \frac{1}{2} e^{2t} - e^t \right) u(t). \]
- The derivative of the step response is:
  \[ y'(t) = \left( e^{2t} - e^t \right) u(t) + \left( \frac{1}{2} + \frac{1}{2} e^{2t} - e^t \right) \delta(t) \]
  \[ = \left( e^{2t} - e^t \right) u(t) + \left( \frac{1}{2} + \frac{1}{2} e^{2t} - e^t \right) \delta(t) \]
  \[ = \left( e^{2t} - e^t \right) u(t). \]
- The step response can also be calculated with a convolution. If \( t > 0 \) then
  \[ y(t) = (h \ast x)(t) = \int_0^t h(\tau) u(t - \tau) \, d\tau \]
  \[ = \int_0^t e^{2\tau} - e^\tau \, d\tau = \frac{1}{2} e^{2\tau} - e^\tau \bigg|_0^t \]
  \[ = \frac{1}{2} (e^{2t} - 1) - e^t + 1 = \frac{1}{2} e^{2t} - e^t + \frac{1}{2}. \]
- If \( t < 0 \) then \( y(t) = 0 \), hence
  \[ y(t) = \left( \frac{1}{2} e^{2t} - e^t + \frac{1}{2} \right) u(t). \]
Example

The system L is described by the differential equation:

\[ y'' - 3y' + 2y = x(t). \]

Find the response to input \( x(t) = 2\delta(t - 1) \).

1. \[ X(s) = 2\mathcal{L}\{\delta(t - 1)\} = 2e^{-s}. \]
2. \[ Y(s) = H(s)X(s) = \frac{2e^{-s}}{s^2 - 3s + 2} \]
   \[ = \frac{2e^{-s}}{s - 2} - \frac{2e^{-s}}{s - 1}. \]
3. \[ y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\left(e^{2t-2} - e^{t-1}\right)u(t - 1). \]

Linear initial value problems and LTI systems

Definition

A \( n\)-th order linear initial value problem is a linear differential equation with initial conditions as follows:

\[
\begin{cases}
  a_n y^{(n)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = G(x(t)), \\
  y(0) = y_0, \\
  y'(0) = y_1, \\
  \vdots \\
  y^{(n-1)}(0) = y_{n-1}.
\end{cases}
\]

with given linear map \( G \) and constants \( y_0, y_1, \ldots, y_{m-1} \).

- If \( y_0 = y_1 = \ldots = y_{m-1} = 0 \), then the initial value problem satisfies the condition of initial rest. In that case the initial value problem is an LTI system.
- If \( y_k \neq 0 \) for some \( 0 \leq k < n \) then the system is not LTI.
If a linear initial value problem is initially not at rest, then there is no transfer function.

We can still use Laplace transforms to solve the initial value problem.

We assume that \( x(t) \) and \( y(t) \) are causal and piecewise smooth.

For the derivatives we use the rule
\[
L\{y^{(k)}\} = sL\{y^{(k-1)}\} - y^{(k-1)}(0^-)
\]
for all \( k = 0 \ldots n - 1 \).

Replace \( y^{(k-1)}(0^-) \) by the initial condition:
\[
y^{(k-1)}(0^-) = y_{k-1}.
\]

Example

Solve the following initial value problem:
\[
\begin{align*}
y'' - y &= 2t, \\
y(0) &= 0, \\
y'(0) &= -2.
\end{align*}
\]

If \( Y = L\{y(t)\} \), then \( L\{y'(t)\} = sY(s) - y(0^-) = sY(s) \).

Subsequently \( L\{y''(t)\} = sL\{y'(t)\} - y'(0^-) = s^2Y(s) + 2 \).

Find the Laplace transform of the differential equation:
\[
(s^2 Y(s) + 2) - Y(s) = 2L\{t\},
\]
\[
(s^2 - 1) Y(s) = \frac{2}{s^2} - 2 = \frac{2(1 - s^2)}{s^2},
\]
\[
Y(s) = \frac{2(1 - s^2)}{s^2(s^2 - 1)} = -\frac{2}{s^2}.
\]

Therefore \( y(t) = L^{-1}\{-2/s^2\} = -2t \).
Example

Solve the following initial value problem:

\[
\begin{align*}
    y' + 2y &= u(t) - u(t - 1), \\
y(0) &= 2.
\end{align*}
\]

- Let \( x(t) = u(t) - u(t - 1) \), then
  \[
  X(s) = \mathcal{L}\{x(t)\} = \frac{1}{s} - \frac{e^{-s}}{s}.
  \]

Example (continued)

- Let \( Y(s) = \mathcal{L}\{y(t)\} \), then
  \[
  \mathcal{L}\{y'(t)\} = s Y(s) - y(0^-) = s Y(s) - 2.
  \]

- The Laplace transform of the differential equation is
  \[
  (s Y(s) - 2) + 2 Y(s) = \frac{1}{s} - \frac{e^{-s}}{s},
  \]
  \[
  (s + 2) Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} + 2,
  \]
  \[
  Y(s) = \frac{1}{s(s + 2)} - \frac{e^{-s}}{s(s + 2)} + \frac{2}{s + 2},
  \]

- We have
  \[
  \frac{1}{s(s + 2)} = \frac{1}{2} \left( \frac{1}{s} - \frac{1}{s + 2} \right) = \frac{1}{2s} - \frac{1}{2(s + 2)}.
  \]
Example (continued)

- \[ Y(s) = \frac{1}{2s} - \frac{1}{2(s+2)} - e^{-s} \left( \frac{1}{2s} + \frac{1}{2(s+2)} \right) + \frac{2}{s+2}. \]
- \[ y(t) = \frac{1}{2} - \frac{1}{2} e^{-2t} - u(t-1) \left( \frac{1}{2} - \frac{1}{2} e^{-2(t-1)} \right) + 2e^{-2t} \]
  \[= \begin{cases} 
  \frac{1}{2} + \frac{3}{2} e^{-2t} & \text{if } 0 \leq t < 1, \\
  \frac{3}{2} e^{-2t} + \frac{1}{2} e^{-2(t-1)} & \text{if } t \geq 1.
  \end{cases} \]

- \[ y(t) \] is not differentiable at 1.

Example

Solve the following initial value problem:
\[
\begin{cases}
  y'' - 3y' + 2y = \delta(t - 2), \\
  y(0) = 0, \\
  y'(0) = 1.
\end{cases}
\]

- Let \( Y(s) = \mathcal{L} \{ y(t) \} \), then
  \[ \mathcal{L} \{ y'(t) \} = s \ Y(s) - y(0^-) = s \ Y(s). \]
  \[ \mathcal{L} \{ y''(t) \} = s^2 \ Y(s) - y'(0^-) = s^2 \ Y(s) - 1. \]
  Take the Laplace transform of the differential equation:
  \[ (s^2 \ Y(s) - 1) - 3s \ Y(s) + 2 \ Y(s) = e^{-2s}. \]
- \[ Y(s) = \frac{1 + e^{-2s}}{s^2 - 3s + 2}. \]
Example (continued)

\[ Y(s) = \frac{1 + e^{-2s}}{s^2 - 3s + 2} \]
\[ = \frac{1}{(s-2)(s-1)} + \frac{e^{-2s}}{(s-2)(s-1)} \]
\[ = \frac{1}{s-2} - \frac{1}{s-1} + e^{-2s}\left(\frac{1}{s-2} - \frac{1}{s-1}\right). \]

\[ y(t) = e^{2t} - e^t + u(t-2)\left(e^{2(t-2)} - e^{t-2}\right). \]

- \( y(t) \) is continuous at 2, but not differentiable.

Example (continued)

Let’s check the answer.

\[ y(t) = e^{2t} - e^t + u(t-2)\left(e^{2(t-2)} - e^{t-2}\right), \]
\[ y'(t) = 2e^{2t} - e^t + \left(2e^{2(t-2)} - e^{t-2}\right)u(t-2), \]
\[ y''(t) = 4e^{2t} - e^t + \left(4e^{2(t-2)} - e^{t-2}\right)u(t-2) + \delta(t-2). \]

Evaluate the differential equation:

\[ y'' - 3y' - 2y = \delta(t-2) \]
Simulation diagrams of LTI systems

- In simulation diagrams LTI systems can be labeled with the transfer function.
- Input and output signals are replaced by the corresponding Laplace transforms.
- The output is the product of input and transfer function.

\[
x(t) \xrightarrow{\int} \int_{-\infty}^{t} x(\tau) d\tau \xrightarrow{X(s)} \frac{1}{s} \xrightarrow{\frac{X(s)}{s}} \frac{H(s)X(s)}{s}
\]

The integrator has transfer function \( H(s) = \frac{1}{s} \).

- Amplifiers with gain \( \alpha \) have transfer function \( H(s) = \alpha \).

Diagram replacement

- The series arrangement of two systems can be replaced by the product of the transfer functions.

\[
H_1(s) \xrightarrow{+} H_2(s) \Rightarrow H_1(s) + H_2(s)
\]

- The series arrangement of two systems can be replaced by the product of the transfer functions.

\[
H_1(s)H_2(s) \Rightarrow H_1(s)H_2(s)
\]
Positive feedback systems

Definition

The system defined by the diagram is called a positive feedback system.

\[ X(s) \rightarrow E(s) \rightarrow H_1(s) \rightarrow Y(s) \]

\[ H_2 Y \]

- Define \( E(s) \) as the output of the adder, then

\[
\begin{align*}
Y &= H_1 E \\
E &= X + H_2 Y
\end{align*}
\]

- Eliminating \( E \) gives \( Y(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}X(s) \).

- The replacement transfer function is

\[
H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}
\]

Eq. 5.7.15

Positive feedback: example

Example

\[ X(t) \rightarrow + \rightarrow \int \rightarrow \alpha \rightarrow Y(t) \]

Find the impulse response of this positive feedback system.

- The replacement transfer function is

\[
H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)} = \frac{1}{s - \alpha}
\]

- The impulse response is \( h(t) = e^{\alpha t}u(t) \).
Negative feedback systems

**Definition**

The system defined by the diagram is called a negative feedback system.

\[ X(s) \rightarrow E(s) \xrightarrow{+} H_1(s) \rightarrow Y(s) \]

\[ H_2 Y \]

- Define \( E(s) \) as the output of the subtractor, then
  \[
  \begin{align*}
  Y &= H_1 E \\
  E &= X - H_2 Y 
  \end{align*}
  \]

- Eliminating \( E \) gives \( Y(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}X(s) \).

- The replacement transfer function is
  \[ H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \]

---

Negative feedback: example

**Example**

\[ x(t) \rightarrow \int \rightarrow y(t) \]

\[ \alpha \]

Find the impulse response of this negative feedback system.

- The replacement transfer function is
  \[
  H(s) = \frac{1}{1 + \frac{1}{s}} = \frac{1}{s + \alpha}.
  \]

- The impulse response is \( h(t) = e^{-\alpha t}u(t) \).
Control systems

- The control system consists of a **controller** with transfer function $H_c(s)$ and a **plant** with transfer function $H(s)$.
- The input signal $r(t)$ is called the **reference signal** and the signal $w(t)$ is the **disturbance**.
- For the **error signal** $e(t)$ we have $e(t) = r(t) - y(t)$.
- The role of the controller is to force the error to zero: $\lim_{t \to \infty} e(t) = 0$.

Example

- In this control system without disturbance the control transfer is $H_c(s) = 2$ and the transfer function of the plant is $H(s) = \frac{1}{s(s + 2)}$.
- For the error $E(s) = \mathcal{L}\{e(t)\}$ we have
  \[
  \begin{cases}
    Y = \frac{2}{s(s + 2)} E \\
    E = R - Y
  \end{cases}
  \]
- Eliminating $E$ gives $Y(s) = \frac{2}{s^2 + 2s + 2}R(s)$.
Example (continued)

- The replacement transfer function is
  \[ H(s) = \frac{2}{s^2 + 2s + 2}. \]
- The Laplace transform of the step response is
  \[ Y(s) = U(s)H(s) = \frac{1}{s} \cdot \frac{2}{s^2 + 2s + 2} = \frac{1}{s} - \frac{1}{(s+1)^2 + 1} - \frac{s+1}{(s+1)^2 + 1}. \]
- The step response is
  \[ y(t) = \left(1 - e^{-t}(\sin t + \cos t)\right)u(t). \]
- Note: \( e(t) = r(t) - y(t) = e^{-t}(\sin t + \cos t) \rightarrow 0 \text{ if } t \rightarrow \infty. \)

Laplace transform of a vector

Definition

The Laplace transform of a vector is calculated termwise. If \( x(t) = (x_1(t), \ldots, x_n(t)) \) and \( X_j = \mathcal{L}\{x_j(t)\} \) then:

\[ X(s) = \mathcal{L}\{x(t)\} = (X_1(s), \ldots, X_n(s)). \]

- Termwise Laplace transform obeys linearity:
  \[ \mathcal{L}\{\alpha x(t) + \beta y(t)\} = \alpha X(s) + \beta Y(s). \]
- For arbitrary matrix \( A \) we have
  \[ \mathcal{L}\{Ax(t)\} = AX(s). \]
- Differentiation rule for termwise Laplace transform:
  \[ \mathcal{L}\{x'(t)\} = sX(s) - x(0^-), \]
  where \( x(0^-) = (x_1(0^-), \ldots, x_n(0^-)). \)
State equations and Laplace transforms

**Recap**  

The state equations for a system are

1. \( v'(t) = Av(t) + x(t)b \)  
2. \( y(t) = c^T v(t) + d x(t) \)

The Laplace transforms of (1) and (2) are

\[
sV(s) - v(0^-) = AV(s) + X(s)b \\
Y(s) = c^T V(s) + d X(s)
\]

In (3), isolate \( V(s) \):

\[
(sI - A)V(s) = v(0^-) + X(s)b \\
V(s) = (sI - A)^{-1}(v(0^-) + X(s)b)
\]

From (4) follows

\[
Y(s) = c^T(sI - A)^{-1}(v(0^-) + X(s)b) + d X(s)
\]

LTI systems

If \( v(0^-) = 0 \), then the system is at initial rest and the system is LTI.

In that case

\[
Y(s) = c^T(sI - A)^{-1}b \cdot X(s) + d X(s) \\
= \left( c^T(sI - A)^{-1}b + d \right) X(s)
\]

The transfer function is

\[
H(s) = c^T(sI - A)^{-1}b + d
\]
Example

Find the impulse response of the system described by

\[
\begin{align*}
v(t) &= \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} v(t) + x(t) \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\
y(t) &= \begin{bmatrix} -1 & 1 \end{bmatrix} v(t) + 2x(t).
\end{align*}
\]

- \[A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad d = 2.\]
- \[sI - A = \begin{bmatrix} s + 3 & -4 \\ 2 & s - 3 \end{bmatrix}.\]
- \[\det(sI - A) = (s + 3)(s - 3) + 8 = s^2 - 1.\]

Example (continued)

\[
(sI - A)^{-1} = \frac{1}{s^2 - 1} \begin{bmatrix} s - 3 & 4 \\ -2 & s + 3 \end{bmatrix}.
\]

- The transfer function is
  \[
  H(s) = \frac{1}{s^2 - 1} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} s - 3 & 4 \\ -2 & s + 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2
  = \frac{2s^2 - 4s - 18}{s^2 - 1} = \frac{6}{s + 1} - \frac{10}{s - 1} + 2.
  \]
- The impulse response is
  \[
  h(t) = \mathcal{L}^{-1} \{H(s)\} = 6e^{-t}u(t) - 10e^t u(t) + 2\delta(t).
  \]
Recap

An LTI system with impulse response \( h(t) \) is BIBO stable if
\[
\int_{-\infty}^{\infty} |h(t)| \, dt < \infty.
\]

- Suppose \( H(s) = \mathcal{L} \{ h(t) \} \) is a rational function that can be decomposed using partial fraction expansion as follows:
\[
H(s) = \sum_{k=1}^{K} \frac{A_k}{(s-a_k)^{n_k}}
\]
with \( a_k \in \mathbb{C} \) and \( n_k \geq 1 \).
- The impulse response is
\[
h(t) = \sum_{k=1}^{K} \frac{A_k}{(n_k-1)!} t^{n_k-1} e^{a_k t} u(t).
\]

We conclude
\[
|h(t)| \leq \sum_{k=1}^{K} \frac{|A_k|}{(n_k-1)!} t^{n_k-1} e^{\text{Re} \, a_k t} u(t).
\]

And consequently
\[
\int_{-\infty}^{\infty} |h(t)| \, dt \leq \sum_{k=1}^{K} \frac{|A_k|}{(n_k-1)!} \int_{0}^{\infty} t^{n_k-1} e^{\text{Re} \, a_k t} \, dt.
\]

Theorem

Let \( H(s) \) be the transfer function of an LTI system. Assume that \( H(s) = N(s)/D(s) \) is rational with \( \deg(N) \leq \deg(D) \) and let \( a_1, \ldots, a_n \) be the zeros of the denominator of \( H(s) \).

- If \( \text{Re} \, a_k < 0 \) for all \( k = 1, \ldots, K \) then all integrals are convergent, and consequently the system is stable.
- If \( \text{Re} \, a_k > 0 \) for \( 1 \leq k \leq K \) then one of the integrals is divergent, and consequently the system is unstable.
Example

- The system of example 5.9.1 has transfer function
  \[
  H(s) = \frac{2s^2 - 4s - 18}{(s + 1)(s - 1)}.
  \]

- The zero \( s = 1 \) lies in the right half-plane, hence the system is unstable.

- Note that
  \[
  h(t) = 6e^{-t}u(t) - 10e^t u(t) + 2\delta(t).
  \]
  It is the term containing \( e^t \) that causes the instability.

- The delta function does not contribute to the instability since
  \[
  \int_{-\infty}^{\infty} |\delta(t)| \, dt = 1.
  \]